Numerical Methods

Locating Roots/Solutions

	Re-arrange first	Find Bounds
Show that the equation $x^3+5x-4=0$ has a solution between $x=0$ and $x=1$.	Show that the equation $2x^3+4x=3$ has a solution between $x=0$ and $x=1$	$f(x)=x^3+2x^2-3x-11$. The equation $f(x)=0$ has one positive root α. Show that $\alpha=2.057$, to 3 decimal places
$x^3+5x-4=0$ Step 1: Make sure you have 0 on one side. $x^3+5x-4=0$ We already have this! Step 2: Plug in both given values into $f(x)=x^3+5x-4$ $f(0)=0^3+5(0)-4=-4$ $f(1)=1^3+5(1)-4=2$ Step 3: State that there is a sign change and the function is continuous One root is -4 and one is 2 therefore sign change (one is plus and one is minus) and $f(x)$ is continuous $\therefore$ solution	$2x^3+4x=3$ Step 1: Make sure you have 0 on one side. We not yet have this and have to re-arrange $2x^3+4x=3$ We have 3 on the right hand side $2x^3+4x-3=0$ We now have this! Step 2: Plug in both values into $f(x)=2x^3+4x-3$. $f(0)=2(0)^3+4(0)-3=-3$ $f(1)=2(1)^3+4(1)-3=3$ Step 3: State that there is a sign change and the function is continuous One root is -3 and one is 3 therefore sign change (one is plus and one is minus) and $f(x)$ is continuous $\therefore$ solution	Step 1: Make sure you have 0 on one side. $f(x)=0$ $x^3+2x^2-3x-11=0$ Step 2: Plug in both values Here we have to use bounds to get the 2 values since we are given the rounded value this time 2.057 has lower bound 2.0565 and upper bound 2.0575 $f(2.0565)=2.0565^3+2(2.0565)^2-3(2.0565)-11=-0.01378$ $f(2.0575)=2.0575^3+2(2.0575)^2-3(2.0575)-11=0.00414$ Step 3: State that there is a sign change and the function is continuous Sign change (one is plus and one is minus) and $f(x)$ is continuous therefore solution

Iteration (aka Fixed Point Iteration)

Notation

A sequence of numbers is formed by the iterative process $u_{n+1}=\frac{3}{u_n+1}$, $u_1=4$ Work out the values of u_2 and u_3	Use the iterative formula $x_{n+1}=\sqrt[3]{x_n^2+2}$ with $x_1=1$, to find α correct to 4 significant figures, showing the result of each iteration.
$u_{n+1}=\frac{3}{u_n+1}$ $u_1=4$ (given starting point) $u_2=\frac{3}{u_1+1}=\frac{3}{4+1}=\frac{3}{5}$ $u_3=\frac{3}{u_2+1}=\frac{3}{\frac{3}{5}+1}=\frac{3}{\frac{8}{5}}=\frac{15}{8}$ **Shortcut on calculator:** Press 4 and then hit enter. Then write $\frac{3}{ans+1}$ Keep pressing enter to get subsequent solutions	Iterative formula $x_{n+1}=\sqrt[3]{(x_n)^2+2}$ $x_1=1$ (given starting point) $x_2=\sqrt[3]{(x_1)^2+2}=\sqrt[3]{1+2}=\sqrt[3]{3}=1.442$ $x_3=\sqrt[3]{(x_2)^2+2}=\sqrt[3]{(1.442)^2+2}=1.598$ $x_4=\sqrt[3]{(x_3)^2+2}=\sqrt[3]{(1.598)^2+2}=1.657$ $x_5=\sqrt[3]{(x_4)^2+2}=\sqrt[3]{(1.657)^2+2}=1.681$ $x_6=\sqrt[3]{(x_5)^2+2}=\sqrt[3]{(1.681)^2+2}=1.690$ $x_7=\sqrt[3]{(x_6)^2+2}=\sqrt[3]{(1.690)^2+2}=1.693$ $x_8=\sqrt[3]{(x_7)^2+2}=\sqrt[3]{(1.693)^2+2}=1.695$ $x_9=\sqrt[3]{(x_8)^2+2}=\sqrt[3]{(1.695)^2+2}=1.695$ $x_{10}=\sqrt[3]{(x_9)^2+2}=\sqrt[3]{(1.695)^2+2}=1.695$ $\therefore \alpha=1.695$ **Shortcut on calculator:** Press 1 and then hit enter. Then write $\sqrt[3]{(\text{ans})^2+2}$ Keep pressing enter to get subsequent solutions

Re-arrangements

Rearrange for one of the x's. This is not like usual where you need the x's on the left hand side and the number on the right hand side.
You may need to factorise first or multiply through first. After rearranging we stick an n+1 on LHS and n on RHS.

Which of the following iteration formulae cannot be found by re-arranging the equation $x^2-9x+2=0$

i. $x_{n+1}=9-\frac{2}{x_n}$
ii. $x_{n+1}=\frac{x_n^2}{9}+\frac{2}{9}$
iii. $x_{n+1}=\frac{9}{2}-\frac{x_n}{2}$
iv. $x_{n+1}=\sqrt{9x_n-2}$

$$x^2-9x+2=0$$

There are 3 ways t re-arrange for x

$x^2-9x+2=0$ $x^2=9x-2$ $x=\sqrt{9x-2}$ $\therefore$ iv. is possible	$x^2-9x+2=0$ $x^2+2=9x$ $9x=x^2+2$ $x=\frac{x^2}{9}+\frac{2}{9}$ $\therefore$ ii. is possible	$x^2-9x+2=0$ Divide through by x first $x-9+\frac{2}{x}=0$ $x=9-\frac{2}{x}$ $\therefore$ i. is possible

iii. is the only iteration formulae which cannot be found by re-arranging the equation $x^2-9x+2=0$

Consider $f(x)=x^3-2x-1$.

i. Show that $x^3-2x-1=0$ has a root between 1 and 2.
ii. Show that $x=\frac{x^3-1}{2}$ is a rearrangement of this equation.
iii. Show that $x=\sqrt[3]{2x+1}$ is a rearrangement of this equation.
iv. Show that $x=\frac{1}{x^2-2}$ is a rearrangement of this equation.
v. Show that $x=\sqrt{2+\frac{1}{x}}$ is a rearrangement of this equation.
vi. Show that $x=\sqrt[4]{x+2x^2}$ is a rearrangement of this equation.
vii. Using $x=\sqrt{2+\frac{1}{x}}$ write down a recurrence relation of the form $x_{n+1}=f(x_n)$ which might be used to solve the equation.
viii. Starting with $x_1=1$, find x_2 and x_3.
ix. Starting with $x_1=1$, use the recurrence relation found in part vi. to find this root to 6dp recording fully the two consecutive iterations which enabled you to stop the procedure.

i.	ii.	iii.	iv.	v.
$x^3-2x-1=0$ $f(x)=x^3-2x-1$ $f(1)=-2$ $(-)$ $f(2)=3$ $(+)$ Sign change and continuous $\therefore$ There is a root between 1 and 2.	$x^3-2x-1=0$ Re-arrange for this x $2x=x^3-1$ $x=\frac{x^3-1}{2}$	$x^3-2x-1=0$ Re-arrange for this x $x^3=2x+1$ $x=\sqrt[3]{2x+1}$	$x^3-2x-1=0$ **Factorise** $x(x^2-2)-1=0$ Now re-arrange for this x $x(x^2-2)-1=0$ $x=\frac{1}{x^2-2}$	$x^3-2x-1=0$ **Factorise** $x^3-2x-1=0$ Now re-arrange for this x $x(x^2-2)-1=0$ $x^2-2=\frac{1}{x}$ $x^2=\frac{1}{x}+2$ $x=\sqrt{2+\frac{1}{x}}$

vi.	vii.	viii.	ix.
$x^3-2x-1=0$ **Multiply through** by x $x^4-2x^2-x=0$ Re-arrange for this x $x^4-2x^2-x=0$ $x^4=2x^2+x$ $x=\sqrt[4]{x+2x^2}$	For the iteration formula we just stick a subscript $n+1$ on the left side and subscript n on the right side of the formula $x=\sqrt{2+\frac{1}{x}}$ $x_{n+1}=\sqrt{2+\frac{1}{x_n}}$ $x_1=1$ $x_2=\sqrt{2+\frac{1}{x_1}}=\sqrt{2+\frac{1}{1}}=1.732$ $x_3=\sqrt{2+\frac{1}{x_2}}=\sqrt{2+\frac{1}{1.732051}}=1.605$	$x_1=1$ $x_2=\sqrt{2+\frac{1}{x_1}}=\sqrt{2+\frac{1}{1}}=1.732051$ $x_3=\sqrt{2+\frac{1}{x_1}}=\sqrt{2+\frac{1}{1.732051}}=1.605412$ $x_4=\sqrt{2+\frac{1}{x_3}}=\sqrt{2+\frac{1}{1.605412}}=1.619535$ $x_5=\sqrt{2+\frac{1}{x_4}}=\sqrt{2+\frac{1}{1.619535}}=1.617857$ $x_6=\sqrt{2+\frac{1}{x_5}}=\sqrt{2+\frac{1}{1.617857}}=1.618055$ $x_7=\sqrt{2+\frac{1}{x_6}}=\sqrt{2+\frac{1}{1.618055}}=1.618032$ $x_8=\sqrt{2+\frac{1}{x_7}}=\sqrt{2+\frac{1}{1.618032}}=1.618034$ $x_9=\sqrt{2+\frac{1}{x_8}}=\sqrt{2+\frac{1}{1.618034}}=1.618034$	We can stop as the last 2 answers were the same to 6 decimal places $\therefore x=1.618034$

Iteration Continued

Re-arrangements and Solving

Re-arranging straight away	Re-arranging straight away	Factorise
i.Show that $2x^3+4x=3$ can be rearranged to give $x=\frac{3}{4}-\frac{x^3}{2}$. ii.Starting with $x_0=0$, use the iteration formula $x_{n+1}=\frac{3}{4}-\frac{x_n^3}{2}$ three times to find an estimate to the solution $2x^3+4x=3$.	i.Show that $x^3-3x^2+3=0$ can be rearranged to give $x=\sqrt[3]{3x^2-3}$. ii.Starting with $x_0=2$ use the iteration formula $x_{n+1}=\sqrt[3]{3x_n^2-3}$ to find the value of x_2 to 3 decimal places.	Show that the equation $x^3+7x-5=0$ has a solution between $x=0$ and $x=1$ i. Show that the equation $x^3+7x-5=0$ can be arranged to give $x=\frac{5}{x^2+7}$. ii. Starting with $x_0=1$, use the iterative formula $x_{n+1}=\frac{5}{x_n^2+7}$ three times to find an estimate for the solution of $x^3+7x-5=0$.
Pick an x to re-arrange for Note: This is not like our usual solving where we get $x's$ on one side and the numbers on the other side. Here we Will have $x's$ on both sides and we need to choose which x to re-arrange for $2x^3+4x-3=0$ $4x=3-2x^3$ $x=\frac{3-2x^3}{4}$ $x=\frac{3}{4}-\frac{2x^3}{4}$ $x=\frac{3}{4}-\frac{x^3}{2}$ **Hack:** We could have cheated/used a trick with this if we couldn't spot how to re-arrange. What write you have at the top, leave a big gap and write what you want at the bottom Work upwards from the bottom (i.e. work backwards from the answer) by expanding, clearing fractions, getting rid of roots etc until you get what you should have started with $2x^3+4x-3=0$ $\uparrow$ $x=\frac{3}{4}-\frac{x^3}{2}$ iii. For the iteration formula we just stick a subscript $n+1$ on the left side and subscript n on the right side of the formula $x=\frac{3}{4}-\frac{x^3}{2}$ Iteration formula: $x_{n+1}=\frac{3}{4}-\frac{x_n^3}{2}$ $x_0=0$ $x_1=\frac{3}{4}-\frac{(x_0)^3}{2}=\frac{3}{4}-0=0.75$ $x_2=\frac{3}{4}-\frac{(x_1)^3}{2}=\frac{3}{4}-\frac{(0.75)^3}{2}=0.539$ $x_3=\frac{3}{4}-\frac{(x_2)^3}{2}=\frac{3}{4}-\frac{(0.539)^3}{2}=0.671$ $\therefore$ Solution of $2x^3+4x=3$ is $x=0.671$	i. Step 1: Make sure you have 0 on one side. $x^3-3x^2+3=0$ Step 2: Pick an x to re-arrange for Note: This is not like our usual solving where we get $x's$ on one side and the numbers on the other side. Here we Will have $x's$ on both sides and we need to choose which x to re-arrange for $x^3-3x^2+3=0$ We can see that we have cube rooted so we have re-arrange for x^3 first $x^3=3+3x^2$ $x=\sqrt[3]{3x^2-3}$ **Hack:** We could have cheated/used a trick with this if we couldn't spot how to re-arrange. What write you have at the top, leave a big gap and write what you want at the bottom Work upwards from the bottom (i.e. work backwards from the answer) by expanding, clearing fractions, getting rid of roots etc until you get what you should have started with. $x^3-3x^2+3=0$ $\uparrow$ $x=\sqrt[3]{3x^2-3}$ ii. For the iteration formula we just stick a subscript $n+1$ on the left side and subscript n on the right side of the formula $x=\sqrt[3]{3x^2-3}$ Iteration formula: $x_{n+1}=\sqrt[3]{3x_n^2-3}$ $x_0=2$ $x_1=\sqrt[3]{3(x_0)^2-3}=\sqrt[3]{3(2)^2-3}=2.0801$ $x_2=\sqrt[3]{3(x_1)^2-3}=\sqrt[3]{3(2.0801)^2-3}=2.153$	$f(x)=x^3+7x-5$ $f(0)=-5$ $f(1)=1+7-5=3$ Sign change and $f(x)$ continuous $\therefore$ At least 1 solution between 0 and 1. i. $x^3+7x-5=0$ $x(x^2+7)-5=0$ $x=\frac{5}{x^2+7}$ ii. $x_0=1$ $x_1=0.625$ $x_2=0.6765$ $x_3=0.6704$ $\therefore$ An estimate for the solution: $x=0.6704$

Factorise First	Factorise First	Factorise First
Show $x^3+2x^2+4=0$ can be re-arranged to give the iterative formula $x_{n+1}=-2-\frac{4}{x_n^2}$. Explain the relationship between the values of x_1, x_2 and x_3 and the equation $x^3+2x^2+4=0$.	Consider $f(x)=x^3-2x-1$. i. Show that $x^3-2x-1=0$ has a root between 1 and 2. ii. Show that $x=\sqrt{2+\frac{1}{x}}$ is a rearrangement of this equation. iii. Hence write down a recurrence relation of the form $x_{n+1}=f(x_n)$ which might be used to solve the equation.	$f(x)=x^3+2x^2-3x-11$. i.Show that $f(x)=0$ can be written as $x=\sqrt{\frac{3x+11}{x+2}}$, $x\neq-2$. ii.Use the iterative formula $x_{n+1}=\sqrt{\frac{3x_n+11}{x_n+2}}$ with $x_1=1$ to find to 3 decimal places the values of x_2, x_3 *and* x_4.
$x^3+2x^2+4=0$ $x^2(x+2)+4=0$ $x^2(x+2)=-4$ Re-arrange for this x $x+2=-\frac{4}{x^2}$ $x=-2-\frac{4}{x^2}$ $\therefore x_{n+1}=-2-\frac{4}{x_n^2}$ The relationship between the values of x_1, x_2 and x_3 and the equation $x^3+2x^2+4=0$ is finding out where the graph of $y=x^3+2x^2+4$ intersects the x axis.	$f(x)=x^3-2x-1$ i. $x^3-2x-1=0$ $f(1)=1-2-1=-2$ $f(2)=8-4-1=3$ $\therefore$ Sign change occurs, therefore there is a root between 1 and 2. ii. Take out x from first 2 terms ONLY and rearrange for x^2 and then root to get x $x(x^2-2)-1=0$ $x(x^2-2)=1$ $x^2-2=\frac{1}{x}$ $x^2=2+\frac{1}{x}$ $x=\sqrt{2+\frac{1}{x}}$ iii. For the iteration formula we just stick a subscript $n+1$ on the left side and subscript n on the right side of the formula $x=\sqrt{2+\frac{1}{x}}$ Recurrence relation: $x_{n+1}=\sqrt{2+\frac{1}{x_n}}$ iv. $x_1=1$ $x_2=\sqrt{2+\frac{1}{x_1}}=\sqrt{2+\frac{1}{1}}=1.732051$ $x_3=\sqrt{2+\frac{1}{x_1}}=\sqrt{2+\frac{1}{1.732051}}=1.605412$ v. $x_1=1$ $x_2=\sqrt{2+\frac{1}{x_1}}=\sqrt{2+\frac{1}{1}}=1.732051$ $x_3=\sqrt{2+\frac{1}{x_1}}=\sqrt{2+\frac{1}{1.732051}}=1.605412$ $x_4=\sqrt{2+\frac{1}{x_3}}=\sqrt{2+\frac{1}{1.605412}}=1.619535$ $x_5=\sqrt{2+\frac{1}{x_4}}=\sqrt{2+\frac{1}{1.619535}}=1.617857$ $x_6=\sqrt{2+\frac{1}{x_5}}=\sqrt{2+\frac{1}{1.617857}}=1.618055$ $x_7=\sqrt{2+\frac{1}{x_6}}=\sqrt{2+\frac{1}{1.618055}}=1.618032$ $x_8=\sqrt{2+\frac{1}{x_7}}=\sqrt{2+\frac{1}{1.618032}}=1.618034$ $x_9=\sqrt{2+\frac{1}{x_8}}=\sqrt{2+\frac{1}{1.618034}}=1.618034$ We can stop as the last 2 answers were the same to 6 decimal places $\therefore x=1.618034$	i. $f(x)=x^3+2x^2-3x-11$ We will NOT re-arrange for this x $x^3+2x^2-3x-11=0$ ii. Instead, we need to re-arrange and factorise x^2 out and re-arrange for x $x^3+2x^2=3x+11$ Factorise $x^2(x+2)=3x+11$ $x^2=\frac{3x+11}{x+2}$ $x=\sqrt{\frac{3x+11}{x+2}}$

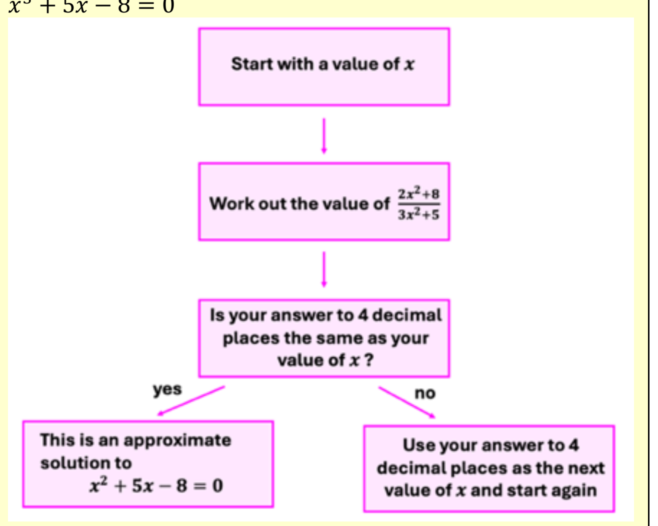
Multiplying through by x first	Turning Point
$x^3+2x^2+x-7=0$ Show that $x_{n+1}=\sqrt[4]{7x_n-x_n^2-2x_n^3}$ is a rearrangement of this equation	$f(x)=3x^4+2x^2-12x+8$ Given that $y=f(x)$ has a single **turning point** at $x=\alpha$, i.Show that α is a solution of the equation $x=\sqrt[3]{1-\frac{x}{3}}$ The iterative formula $x_{n+1}=\sqrt[3]{1-\frac{x_n}{3}}$ is used with $x_1=1$ to find an approximate value for α. ii.Calculate the value of x_2 and the value of x_3, giving each answer to 4 decimal places. iii.Using a suitable interval and a suitable function that should be stated, show that to 3 decimal places α is 0.889
	$f(x)=3x^4+2x^2-12x+8$ i. **turning point** hence we need to differentiate first and set equal to zero $f'(x)=12x^3+4x-12$ When $f'(x)=0$, $12x^3+4x-12=0$ Re-arrange for this x $3x^3+x-3=0$ $3x^3=3-x$ $x^3=1-\frac{x}{3}$ $x=\sqrt[3]{1-\frac{x}{3}}$ ii. $x_2=0.8736$ $x_3=0.8916$ $x_4=0.8891$ $x_5=0.8894$ iii. We choose 0.8885 and 0.8895. The equation is $f'(x)=12x^3+4x-12$. $f'(0.8885)=-0.029$ $f'(0.8895)=0.0034$ Sign change and $f'(x)$ is continuous. So $\alpha=0.889$.

Iteration Continued

Interpretation Of Solutions

i.Show $x^3+2x^2+4=0$ can be re-arranged to give the iterative formula $x_{n+1}=-2-\frac{4}{x_n^2}$. ii.Explain the relationship between the values of x_1, x_2 and x_3 and the equation $x^3+2x^2+4=0$.	i.Show $3x^2-x^3+3=0$ can be re-arranged to give $x=3+\frac{3}{x^2}$. ii.Using $x_{n+1}=3+\frac{3}{x_n^2}$ with $x_0=3.2$,find the values of x_1, x_2 and x_3 iii.Explain what the values x_1, x_2 and x_3 represent	Using $x_n=-2-\frac{4}{x_n^2}$ with $x_0=-2.5$ i. Find the values of x_1, x_2 and x_3 to 3 d.p. ii. Explain the relationship between the values of x_1, x_2 and x_3 and the equation $x^3+2x^2+4=0$
i. $x^3+2x^2+4=0$ We re-arrange for this x $x^3=-2x^2-4$ Divide through by x^2 $x=-2-\frac{4}{x^2}$ $\therefore x_{n+1}=-2-\frac{4}{x_n^2}$ i. The relationship between the values of x_1, x_2 and x_3 and the equation $x^3+2x^2+4=0$ is finding out where the graph of $y=x^3+2x^2+4$ intersects/crosses the x axis i.e. the roots of the graph of x^3+2x^2+4. Being a bit more detailed we say x_1, x_2 and x_3 are the increasingly accurate estimations of the solutions $x^3+2x^2+4=0$		i. $x_0=-2.5$ $x_1=-2-\frac{4}{x_0^2}=-2-\frac{4}{(-2.5)^2}=-2.64$ $x_2=-2-\frac{4}{x_1^2}=-2-\frac{4}{(-2.64)^2}=-2.574$ $x_3=-2-\frac{4}{x_2^2}=-2-\frac{4}{(-2.574)^2}=-2.604$ ii. $x=-2-\frac{4}{x^2}$ is a re-arrangement of $x^3+2x^2+4=0$ x_1, x_2, and x_3 are the increasingly accurate estimations of the solution for $x^3+2x^2+4=0$ i.e. the roots of the graph $x^3+2x^2+4=0$ (where $x^3+2x^2+4=0$ crosses the x axis
Using $x_{n+1}=-3-\frac{2}{x_n^2}$, with $x_0=-3.5$ i. Find the values of x_1, x_2 and x_3 to 3 decimal places ii. Explain the relationship between the values of x_1, x_2 and x_3 and the equation $x^3+3x^2+2=0$	i. Show that the equation $20-x^3-7x^2=0$ can be re-arranged to give $x=\frac{20}{x^2}-7$ ii. Using $x_{n+1}=\frac{20}{x_n^2}-7$ with $x_0=-9$ Find the values of x_1, x_2 and x_3 iii. Explain what the values x_1, x_2 and x_3 represent	i. Show that the equation $x^3+7x-5=0$ has a solution between $x=0$ and $x=1$ ii. Show that the equation $x^3+7x-5=0$ can be re-arranged to give $x=\frac{5}{x^2+7}$ iii. Starting with $x_0=1$ use the iteration formula $x_{n+1}=\frac{5}{x_n^2+7}$ three times to find an estimate for the solution to $x^3+7x-5=0$ iv. By substituting your answer from part iii. into x^3+7x-5 comment on the accuracy of your estimate to the solution to $x^3+7x-5=0$
i. $x_0=-3.5$ $x_1=-3-\frac{2}{x_0^2}=-3-\frac{2}{(-3.5)^2}=-3.163$ $x_2=-3-\frac{2}{x_1^2}=-3-\frac{2}{(-3.163)^2}=-3.200$ $x_3=-3-\frac{2}{x_2^2}=-3-\frac{2}{(-3.200)^2}=-3.195$ ii. $x=-3-\frac{2}{x^2}$ is a re-arrangement of $x^3+3x^2+2=0$ x_1, x_2, and x_3 are the increasingly accurate estimations of the solution for $x^3+3x^2+2=0$ i.e. the roots of the graph $x^3+3x^2+2=0$ (where $x^3+3x^2+2=0$ crosses the x axis)	$20-x^3-7x^2=0$ **Way 1: Divide all terms by x^2** $20-7x^2=x^3$ $\frac{20}{x^2}-7=x$ $x=\frac{20}{x^2}-7$ **Way 2: factorise out x** $20-7x^2=x^3$ $20-7x^2=xx^2$ Re-arrange for this x $x=\frac{20-7x^2}{x^2}$ $x=\frac{20}{x^2}-\frac{7x^2}{x^2}$ $x=\frac{20}{x^2}-7$ iii. x_1, x_2, and x_3 are the increasingly accurate estimations of the solution for $20-x^3-7x^2=0$	i. $f(x)=x^3+7x-5$ $f(0)=-5$ $f(1)=1+7-5=3$ Sign change and $f(x)$ continuous $\therefore$ At least 1 solution between 0 and 1. ii. $x^3+7x-5=0$ $x(x^2+7)-5=0$ $x=\frac{5}{x^2+7}$ iii. $x_0=1$ $x_1=\frac{5}{x_0^2+7}=\frac{5}{(-3.5)^2+7}=0.625$ $x_2=\frac{5}{x_1^2+7}=\frac{5}{(0.625)^2+7}=0.6765$ $x_3=\frac{5}{x_2^2+7}=\frac{5}{(0.6765)^2+7}=0.6704$ $\therefore$ An estimate for the solution: $x=0.6704$ iv. $(0.6704)^3+7(0.6705)-5=-0.005898$ Solution is relatively close to 0 so can be considered a close estimate

The following iterative process can be used to find solutions to $x^3+5x-8=0$



i. Use this iterative process to find a solution to 4 decimal places of $x^3+5x-8=0$. Start with the value $x=1$
ii. By substituting your answer into x^3+5x-8 comment on the accuracy of your solution to $x^3+5x-8=0$

i.
We keep pressing enter until our answer is the same to 4 decimal places

$x_0=1$

$x_1=\frac{2(x_0^3)+8}{(x_0^2)+5}=\frac{2(1^3)+8}{(1)^2+5}=1.25$

$x_2=\frac{2(x_1^3)+8}{(x_1^2)+5}=\frac{2(1.25^3)+8}{(1.25)^2+5}=1.229032258$

$x_3=\frac{2(x_2^3)+8}{3(x_2^2)+5}=\frac{2(1.229032258^3)+8}{(1.229032258)^2+5}=1.228860251$

1.229032258 and 1.228860251 are the same to 4 decimal places

$\therefore$ An estimate for the solution:
$x=1.2289$

ii.
$1.2289^3+5(1.2289)-8=0.000379$
This is very close to 0, making 1.2289 a very accurate solution for $x^3+5x-8=0$

Note:
$1.25^3+5(1.25)-8=0.2031$
$1.229032258^3+5(1.229032258)-8=0.0016$
$1.228860251^3+5(1.228860251)-8=0.0000$
Notice how the more accurate the solution, the closer to 0 (they are increasingly accurate estimates)

Below are three iteration formulae to find approximate solutions to the equation $6x-x^2-7=0$
Also shown are three possible values for x_0

A	$x_{n+1}=\frac{x_n^2}{6}+\frac{7}{6}$	$x_0=1.2$	**1**
B	$x_{n+1}=\sqrt{6x_n-7}$	$x_0=0$	**2**
C	$x_{n+1}=6-\frac{7}{x_n}$	$x_0=2$	**3**

Match each iterative formula to a suitable so that each formula gives an approximate solution to the equation $6x-x^2-7=0$. You may only use each value of x_0 once

Checking A:
$x_{n+1}=\frac{x_n^2}{6}+\frac{7}{6}$
$x_1=\frac{0}{6}+\frac{7}{6}=\frac{7}{6}$
A $\Rightarrow$ *2*

Checking B:
B is not possible for *2* since when $x_0=0$, $x_1=\sqrt{6\times0-7}=\sqrt{-7}$ which is not possible

Checking C:
C is not possible for *2* as when $x_0=0$, $x_1=6-\frac{7}{0}$ which is not possible since can't divide by 0

B $\Rightarrow$ *3*
Since *B* is not possible for *1* as when $x_0=1.2$, $x_1=\sqrt{6\times1.2-7}=\sqrt{0.2}$

$x_2=\sqrt{6\times0.4472\ldots-7}=\sqrt{-4.316\ldots}$ which is not possible

C $\Rightarrow$ *1*

Worded

The number of slugs in a garden t days from now is p_t, where

$$P_0 = 100$$
$$P_{t+1} = 1.06p_t$$

Work out the number of slugs in the garden 3 days from now

$$P_0 = 100$$

$$P_1 = 1.06P_0 = 1.06(100) = 106$$

$$P_2 = 1.06P_1 = 1.06(106) = 112.36$$

$$P_3 = 1.06P_2 = 1.06(112.36)$$
$$= 119.1016 \approx 119 \text{ slugs}$$

The number of animals in a population at the start of year t is P_t

The number of animals at the start of year 1 is 400

Given that

$$P_{t+1} = 1.01P_t$$

Work out the number of animals at the start of year 3

$$P_1 = 400$$

$$P_2 = 1.01P_1 = 1.01(30) = 404$$

$$P_3 = 1.01P_2 = 1.01(404) = 409.04$$
$$\approx 408 \text{ animals}$$

At the start of year n, the quantity of a radioactive metal is P_N

At the start of the following year, the quantity of the same metal is given by

$$P_{n+1} = 0.87P_n$$

At the start of 2016 there were 30 grams of the metal. What will be the quantity of the metal at the start of 2019?

$$P_{2016} = 30$$

$$P_{2017} = 0.87P_{2016} = 0.87(30) = 26.1$$

$$P_{2018} = 0.87P_{2017} = 0.87(26.1)$$
$$= 22.707$$

$$P_{2019} = 0.87P_{2018} = 0.87(22.707)$$
$$= 19.755 \approx 20 \text{ grams}$$

The number of bees in a beehive at the start of year n is P_n

The number of bees in a beehive at the start of the following year is given by

$$P_{n+1} = 1.05(P_n - 250)$$

At the start of 2015 there were 9500 bees in the beehive

How many bees will there be in the beehive at the start of 2018?

At the beginning of 2015, there were 9500 bees

This tells us that

$$P_{2015} = 9500$$

We are given that

$$P_{n+1} = 1.05(P_n - 250)$$

We can use the fact that

$$P_{2016} = 1.05(P_{2015} - 250)$$
$$P_{2017} = 1.05(P_{2016} - 250)$$
$$P_{2018} = 1.05(P_{2017} - 250)$$

$$\mathbf{P_{2016} = 1.05(P_{2015} - 250)}$$
$$P_{2016} = 1.05(9500 - 250) = 9712.5$$

$$\mathbf{P_{2017} = 1.05(P_{2016} - 250)}$$
$$P_{2017} = 1.05(9712.5 - 250) = 9935.625$$

$$\mathbf{P_{2018} = 1.05(P_{2017} - 250)}$$
$$P_{2018} = 1.05(9935.625 - 250) = 10170$$

10,170 bees

The number of moose in Alaska at the start of year n is P_n

The number of moose in Alaska at the start of the following year is given by $P_{n+1} = 1.04\ (P_n - G)$

where G is a constant

At the beginning of 2015, there were 200 000 moose in Alaska.

At the beginning of 2016, there were 200 720 moose in Alaska.

Work out how many moose there were in Alaska at the beginning of 2017

At the beginning of 2015, there were 200 000 moose in Alaska.

At the beginning of 2016, there were 200 720 moose in Alaska.

This tells us that

$$P_{2015} = 200{,}000$$
$$P_{2016} = 200{,}720$$

Firstly, we use the information to find G

$$P_{2016} = 1.04(P_{2015} - G)$$

$$200{,}720 = 1.04(200{,}000 - G)$$

$$200{,}000 - G = \frac{200{,}720}{1.04}$$

$$G = 200{,}000 - \frac{200{,}720}{1.04} = 7000$$

Sub this value found back into

$$P_{n+1} = 1.04\ (P_n - G)$$

$$P_{n+1} = 1.04(P_n - 7000)$$

$$P_{2017} = 1.04(P_{2016} - 7000)$$

$$= 1.04(200720 - 7000) = 201177.6$$

$$\approx 201178 \text{ moose}$$

`At the start of year n, the number of animals in a population is P_n

At the start of the following year, the number of animals in the population is P_{n+1}, where

$$P_{n+1} = kP_n$$

At the start of 2017 the number of animals in the population was 4000

At the start of 2019 the number of animals in the population was 3610

Find the value of the constant k

At the start of 2017 the number of animals in the population was 4000

At the start of 2019 the number of animals in the population was 3610

This tells us that

$$P_{2017} = 4000$$
$$P_{2019} = 3610$$

2017 – 2018:

$$P_{2018} = kP_{2017}$$

$$P_{2018} = k(4000)$$

$$P_{2018} = 4000k$$

2018 – 2019:

$$P_{2019} = kP_{2018}$$

$$3610 = k4000k$$

$$3610 = 4000k^2$$

$$4000k^2 = 3610$$

$$k = \sqrt{\frac{3610}{4000}} = 0.95$$

Worded Harder

A hot air balloon is descending.

The height of the balloon n minutes after it starts to descend is h metres

The height of the balloon $(n+1)$ minutes after it starts to descend, h_{n+1} metres, is given by

$$h_{n+1} = K \times h_n + 20 \text{ where } K \text{ is a constant}$$

The balloon starts to descend from a height of 1200 metres at 09:15

At 09:16 the height of the balloon is 1040 metres

Work out the height of the balloon at 09:18

$$h_{n+1} = K \times h_n + 20$$

$$1040 = 1200k + 20$$
$$k = 0.85$$

$$h_{n+1} = 0.85 \times h_n + 20$$

9:17:

$$0.85(1040) + 20 = 904$$

9:18

$$0.85(904) + 20 = 788.4\ m$$

At time $t = 0$ hours a tank is full of water

Water leaks from a tank

At the end of every hour there is 2% less water in the tank than at the start of the hour

The volume of water, in litres, in the tank at time t hours is V_t

Given that

$$V_0 = 2000$$
$$V_{t+1} = kV_t$$

Write down the value of k

Way 1: %age Multiplier Way

$$V_{t+1} = kV_t$$

$$V_{t+1} = \left(1 - \frac{2}{100}\right)V_t$$

$$V_{t+1} = 0.98V_t$$

$$k = 0.98$$

Way 2:

$$0.02(2000) = 40$$

$$2000 - 40 = 1960$$

$$1960 = k(2000)$$

$$k = \frac{1960}{2000} = 0.98$$

$$k = 0.98$$

At the start of year n, the population of a species is P_n

At the start of the following year, the population of a species is given by

$$P_{n+1} = kP_n, \text{ where k is a positive constant}$$

The population of the species at the start of year 1 is 8 million

The population of the species at the start of year 2 is 6 million

i. Work out the population of the species at the start of year 3

At the start of year 5 the value of k is increased by 0.3 to a new constant value.

Louise thinks that from the start of year 5 the population of the species would increase year on year.

i. Is Louise correct? You must give a reason for your answer.

The population of the species at the start of year 1 is 8 million

The population of the species at the start of year 2 is 6 million

This tells us that

$$P_1 = 8000000$$
$$P_2 = 6000000$$

i.

$$P_{n+1} = kP_n$$

$$P_2 = kP_1$$

$$6000000 = k(8000000)$$
$$k = \frac{6}{8} = 0.75$$
$$P_{n+1} = 0.75P_n$$
$$P_3 = 0.75P_2$$
$$= 0.75(6000000) = 4{,}500{,}000 = 4.5 \text{ million}$$

ii.

$$P_3 = 0.75P_2$$
$$0.75 + 0.3 = 1.0$$
$$P_3 = 1.05P_2$$

Yes, Louise is correct. The population will begin increasing each year by 5% as each year it is multiplied by 1.05 which is greater than 1.

The number of insects in a population at the start of year n is P_n

The number of insects in the population at the start of year $(n+1)$ is P_{n+1} where

$$P_{n+1} = kP_n$$

Given that k has a constant value of 1.13

ii. Find out how many years it takes for the number of insects in the population to double. You must show how you get your answer.

The value of k actually increases year on year from its value of 1.13 in year 1

How does this affect your answer to part a?

i.

Way 1:

$$P_{n+1} = kP_n$$
$$P_2 = kP_2$$

$$P_3 = kP_2 = k(kP_1) = k^2P_1$$

$$P_4 = kP_3 = k(k^2P_1) = k^3P_1$$

$$P_5 = kP_4 = k(k^3P_1) = k^4P_1$$

etc

Then you can see you're just increasing the power of the original k each time and the principal is the same. Hence it mimics compounds interest.

Way 2:

$$P_{n+1} = kP_n$$

1st year: $1.13P_n$

2nd year: 1.13^2P_n

See when gets to $2P_n$

$$1.13^xP_n = 2P_n$$

$$1.13^x = 2$$

Use trial and error to find $\boldsymbol{x}$

$$x = 6$$

Way 3: Realise that this is just compound interest

$$P_{n+1} = kP_n$$

This is no different to the compound interest formula

$$P_{n+1} = kP_n$$

$$\text{new amount} = \textbf{amount}(1 \pm \%)^t$$

$$2A = A(1 \pm \%)^t$$

$$2 = (1.13)^t$$

Use trial and error to find t

$$t = 6$$

It takes 6 years

ii.

If the value of k increases each year, the growth rate of the insect population will accelerate. It will take less time to double the quantity.